A Primer on Strategic Games

Krzysztof R. Apt

(so *not* Krzystof and *definitely not* Krystof)

CWI, Amsterdam, the Netherlands,

University of Amsterdam

Overview

- Best response,
- Nash equilibrium,
- Weak/strict dominance,
- Iterated elimination of strategies,
- Mixed strategies,
- Variations on the definition,
- Pre-Bayesian games,
- Mechanism design: implementation in dominant strategies.

Strategic Games: Definition

Strategic game for $n \geq 2$ players:

- (possibly infinite) set S_i of strategies,
- payoff function $p_i : S_1 \times \ldots \times S_n \rightarrow \mathbb{R}$,

for each player i.

Basic assumptions:

- players choose their strategies simultaneously,
- each player is rational: his objective is to maximize his payoff,
- players have common knowledge of the game and of each others' rationality.

Three Examples

Prisoner's Dilemma

	C	D
C	$2, 2$	$0, 3$
D	$3, 0$	$1, 1$

The Battle of the Sexes

	F	B
F	$2, 1$	$0, 0$
B	$0, 0$	$1, 2$

Matching Pennies

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

Three Main Concepts

- Notation: $s_i, s'_i \in S_i$, $s, s', (s_i, s_{-i}) \in S_1 \times \ldots \times S_n$.
- s_i is a best response to s_{-i} if

$$\forall s'_i \in S_i \; p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i}).$$

- s is a Nash equilibrium if $\forall i \;\; s_i$ is a best response to s_{-i}:

$$\forall i \in \{1, \ldots, n\} \; \forall s'_i \in S_i \; p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i}).$$

  Intuition: In a Nash equilibrium no player can gain by *unilaterally* switching to another strategy.
- s is Pareto efficient if for no s'

$$\forall i \in \{1, \ldots, n\} \; p_i(s') \geq p_i(s),$$
$$\exists i \in \{1, \ldots, n\} \; p_i(s') > p_i(s).$$

Nash Equlibrium

Prisoner's Dilemma: 1 Nash equlibrium

	C	D
C	$2, 2$	$0, 3$
D	$3, 0$	$1, 1$

The Battle of the Sexes: 2 Nash equlibria

	F	B
F	$2, 1$	$0, 0$
B	$0, 0$	$1, 2$

Matching Pennies: no Nash equlibrium

	H	T
H	$1, -1$	$-1, 1$
T	$-1, 1$	$1, -1$

Dominance

- s'_i is strictly dominated by s_i if

$$\forall s_{-i} \in S_{-i} \; p_i(s_i, s_{-i}) > p_i(s'_i, s_{-i}),$$

- s'_i is weakly dominated by s_i if

$$\forall s_{-i} \in S_{-i} \; p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i}),$$
$$\exists s_{-i} \in S_{-i} \; p_i(s_i, s_{-i}) > p_i(s'_i, s_{-i}).$$

Prisoner's Dilemma

	C	D
C	$2, 2$	$0, 3$
D	$3, 0$	$1, 1$

Why a dilemma?

- (D, D) is the unique Nash equilibrium,
- For each player C is strictly dominated by D,
- (C, C) is a Pareto efficient outcome in which each player has a $>$ payoff than in (D, D).

Prisoner's Dilemma for n Players

Assume $k_i(n-1) > l_i > 0$ for all i.

$$p_i(s) := \begin{cases} k_i|s_{-i}(C)| + l_i & \text{if } s_i = D \\ k_i|s_{-i}(C)| & \text{if } s_i = C. \end{cases}$$

- For $n = 2, k_i = 2$ and $l_i = 1$ we get the original Prisoner's Dilemma game.
- $p_i(C^n) = k_i(n-1) > l_i = p_i(D^n)$,
  so for all players C^n yields a $>$ payoff than D^n.
- For all players strategy C is strictly dominated by D:
  $p_i(D, s_{-i}) - p_i(C, s_{-i}) = l_i > 0$.

Quiz

	H	T	E
H	$1, -1$	$-1, \ 1$	$-1, -1$
T	$-1, \ 1$	$1, -1$	$-1, -1$
E	$-1, -1$	$-1, -1$	$-1, -1$

- What are the Nash equilibria of this game?

Answer

	H	T	E
H	$1,-1$	$-1,\ \ 1$	$-1,-1$
T	$-1,\ \ 1$	$1,-1$	$-1,-1$
E	$-1,-1$	$-1,-1$	$-1,-1$

- (E, E) is the only Nash equilibrium.
- It is a Nash equilibrium in weakly dominated strategies.

IESDS: Example 1

	L	M	R
T	$3, 0$	$2, 1$	$1, 0$
C	$2, 1$	$1, 1$	$1, 0$
B	$0, 1$	$0, 1$	$0, 0$

- B is strictly dominated by T,
- R is strictly dominated by M.

By eliminating them we get:

	L	M
T	$3, 0$	$2, 1$
C	$2, 1$	$1, 1$

IESDS, Example 1ctd

	L	M
T	$3, 0$	$2, 1$
C	$2, 1$	$1, 1$

Now C is strictly dominated by T, so we get:

	L	M
T	$3, 0$	$2, 1$

Now L is strictly dominated by M, so we get:

	M
T	$2, 1$

We solved the game by IESDS.

IESDS

Theorem

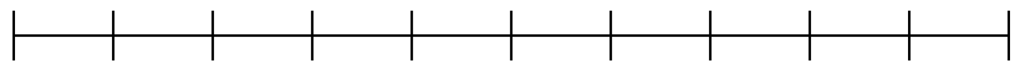
- If G' is an outcome of IESDS starting from a finite G, then s is a Nash equilibrium of G' iff it is a Nash equilibrium of G.
- If G is finite and is solved by IESDS, then the resulting joint strategy is a unique Nash equilibrium of G.
- (Gilboa, Kalai, Zemel, '90) Outcome of IESDS is unique (order independence).

IESDS: Example

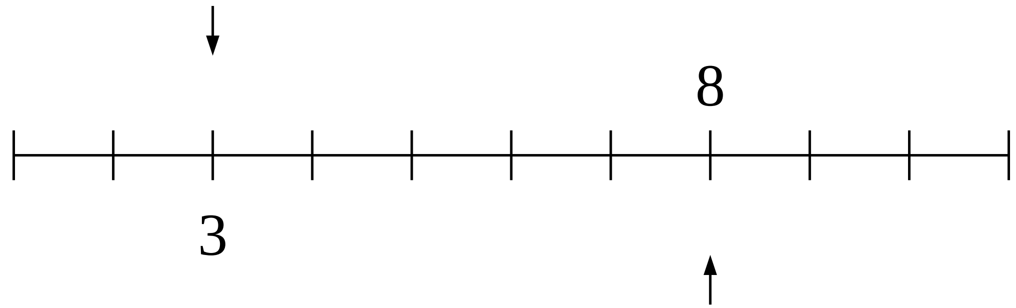
Location game (Hotelling '29)

- 2 companies decide simultaneously their location,
- customers choose the closest vendor.

Example: Two bakeries, one (discrete) street.

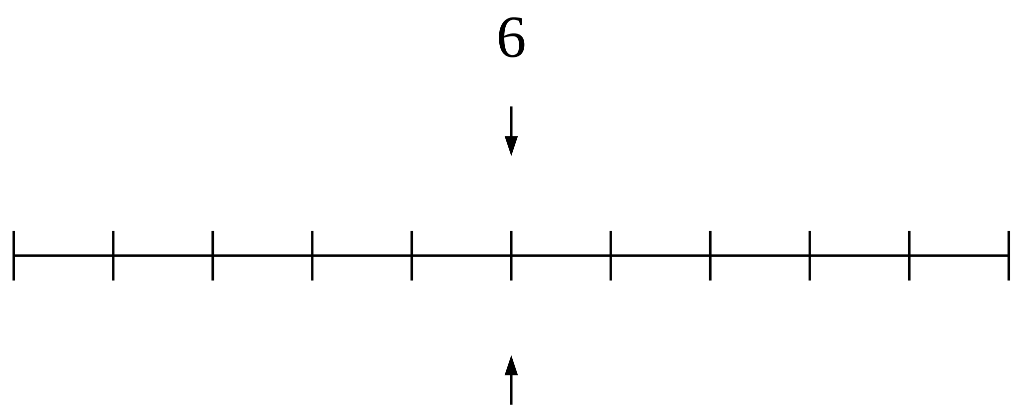


For instance:



Then baker$_1(3, 8) = 5$, baker$_2(3, 8) = 6$.
Where do I put my bakery?

Answer



Then:
baker$_1(6,6) = 5.5$,
baker$_2(6,6) = 5.5$.

- $(6,6)$ is the outcome of IESDS.
- Hence $(6,6)$ is a unique Nash equilibrium.

IEWDS

Theorem

- If G' is an outcome of IEWDS starting from a finite G and s is a Nash equilibrium of G', then s is a Nash equilibrium of G.
- If G is finite and is solved by IEWDS, then the resulting joint strategy is a Nash equilibrium of G.
- Outcome of IEWDS does not need to be unique (no order independence).

IEWDS: Example 1

Beauty-contest game (Moulin, '86)

- each set of strategies = $\{1, \ldots, 100\}$,
- payoff to each player:
  1 is split equally between the players whose submitted number is closest to $\frac{2}{3}$ of the average.

Example
submissions: $29, 32, 29$; average: 30,
payoffs: $\frac{1}{2}, 0, \frac{1}{2}$.

- This game is solved by IEWDS.
- Hence it has a Nash equilibrium, namely $(1, \ldots, 1)$.

IEWDS: Example 2

The following game has two Nash equilibria:

	X	Y	Z
A	$2, 1$	$0, 1$	$1, 0$
B	$0, 1$	$2, 1$	$1, 0$
C	$1, 1$	$1, 0$	$0, 0$
D	$1, 0$	$0, 1$	$0, 0$

- D is weakly dominated by A,
- Z is weakly dominated by X.

By eliminating them we get:

	X	Y
A	$2, 1$	$0, 1$
B	$0, 1$	$2, 1$
C	$1, 1$	$1, 0$

Example 2, ctd

	X	Y
A	$2,1$	$0,1$
B	$0,1$	$2,1$
C	$1,1$	$1,0$

Next, we get

	X
A	$2,1$
B	$0,1$
C	$1,1$

and finally

	X
A	$2,1$

IEWDS: Example 3

	L	R
T	$1, 1$	$1, 1$
B	$1, 1$	$0, 0$

can be reduced both to

	L	R
T	$1, 1$	$1, 1$

and to

	L
T	$1, 1$
B	$1, 1$

Infinite Games

Consider the game with

- $S_i := \mathbb{N}$,
- $p_i(s) := s_i$.

Here

- every strategy is strictly dominated,
- in one step we can eliminate
  - all strategies,
  - all $\neq 0$ strategies,
  - one strategy per player.

Infinite Games (2)

Conclusions For infinite games

- IESDS is not order independent,
- definition of order independence has to be modified.

IENBR: Example 1

	X	Y
A	$2, 1$	$0, 0$
B	$0, 1$	$2, 0$
C	$1, 1$	$1, 2$

- No strategy strictly or weakly dominates another one.
- C is never a best response.

Eliminating it we get

	X	Y
A	$2, 1$	$0, 0$
B	$0, 1$	$2, 0$

from which in two steps we get

	X
A	$2, 1$

IENBR

Theorem

- If G' is an outcome of IENBR starting from a finite G, then s is a Nash equilibrium of G' iff it is a Nash equilibrium of G.
- If G is finite and is solved by IENBR, then the resulting joint strategy is a unique Nash equilibrium of G.
- (Apt, '05) Outcome of IENBR is unique (order independence).

IENBR: Example 2

Location game on the open real interval $(0, 100)$.

$$p_i(s_i, s_{3-i}) := \begin{cases} s_i + \frac{s_{3-i} - s_i}{2} & \text{if } s_i < s_{3-i} \\ 100 - s_i + \frac{s_i - s_{3-i}}{2} & \text{if } s_i > s_{3-i} \\ 50 & \text{if } s_i = s_{3-i} \end{cases}$$

- No strategy strictly or weakly dominates another one.
- Only 50 is a best response to some strategy (namely 50).
- So this game is solved by IENBR, in one step.

Mixed Extension of a Finite Game

- Probability distribution over a finite non-empty set A:

$$\pi : A \rightarrow [0, 1]$$

  such that $\sum_{a \in A} \pi(a) = 1$.
- Notation: ΔA.

Fix a finite strategic game $G := (S_1, \ldots, S_n, p_1, \ldots, p_n)$.

- Mixed strategy of player i in G: $m_i \in \Delta S_i$.
- Joint mixed strategy: $m = (m_1, \ldots, m_n)$.

Mixed Extension of a Finite Game (2)

- Mixed extension of G:

$$(\Delta S_1, \ldots, \Delta S_n, p_1, \ldots, p_n),$$

where

$$m(s) := m_1(s_1) \cdot \ldots \cdot m_n(s_n)$$

and

$$p_i(m) := \sum_{s \in S} m(s) \cdot p_i(s).$$

- Theorem (Nash '50) Every mixed extension of a finite strategic game has a Nash equilibrium.

Kakutani's Fixed Point Theorem

Theorem (Kakutani '41)
Suppose A is a compact and convex subset of $\mathbb{R}^n$ and

$$\Phi : A \rightarrow \mathcal{P}(A)$$

is such that

- $\Phi(x)$ is non-empty and convex for all $x \in A$,
- for all sequences (x_i, y_i) converging to (x, y)

  $$y_i \in \Phi(x_i) \text{ for all } i \geq 0,$$

  implies that

  $$y \in \Phi(x).$$

Then $x^* \in A$ exists such that $x^* \in \Phi(x^*)$.

Proof of Nash Theorem

Fix $(S_1, \ldots, S_n, p_1, \ldots, p_n)$. Define

$$best_i : \Pi_{j \neq i} \Delta S_j \rightarrow \mathcal{P}(\Delta S_i)$$

by

$best_i(m_{-i}) := \{m'_i \in \Delta S_i \mid p_i(m'_i, m_{-i})$ attains the maximum$\}$.

Then define

$$best : \Delta S_1 \times \ldots \Delta S_n \rightarrow \mathcal{P}(\Delta S_1 \times \ldots \times \Delta S_n)$$

by

$$best(m) := best_1(m_{-1}) \times \ldots \times best_1(m_{-n}).$$

Note m is a Nash equilibrium iff $m \in best(m)$.
$best(\cdot)$ satisfies the conditions of Kakutani's Theorem.

Comments

- First special case of Nash theorem: Cournot (1838).
- Nash theorem generalizes von Neumann's Minimax Theorem ('28).
- An alternative proof (also by Nash) uses Brouwer's Fixed Point Theorem.
- Search for conditions ensuring existence of Nash equilibrium.

2 Examples

Matching Pennies

	H	T
H	$1, -1$	$-1, \quad 1$
T	$-1, \quad 1$	$1, -1$

- $(\frac{1}{2} \cdot H + \frac{1}{2} \cdot T, \frac{1}{2} \cdot H + \frac{1}{2} \cdot T)$ is a Nash equilibrium.
- The payoff to each player in the Nash equilibrium: 0.

The Battle of the Sexes

	F	B
F	$2, 1$	$0, 0$
B	$0, 0$	$1, 2$

- $(2/3 \cdot F + 1/3 \cdot B, 1/3 \cdot F + 1/3 \cdot B)$ is a Nash equilibrium.
- The payoff to each player in the Nash equilibrium: $2/3$.

Variations on the Definition

- Strategic games with qualitative preferences
(Osborne, Rubinstein '94)
$(S_1, \ldots, S_n, \succeq_1, \ldots, \succeq_n)$, where each $\succeq_i$ is a preference relation on $S_1 \times \ldots \times S_n$.
- Strategic games with parametrized preferences
(Apt, Rossi, Venable '08)
Each player i has a set of strategies S_i and a preference relation $\succeq(s_{-i})$ on S_i parametrized by $s_{-i} \in S_{-i}$.
- Conversion/preference games
(Le Roux, Lescanne, Vestergaard '08)
The game consists of a set S of situations and for each player i a preference relation $\succeq_i$ on S and a conversion relation $\rightarrow_i$ on S.

Graphical Games

(Kearns, Littman, Singh '01)

- Each player i has a set of neighbours *neigh*(i).
- Payoff for player i is a function

$$p_i : \times_{j \in \textit{neigh}(i) \cup \{i\}} S_j \longrightarrow \mathbb{R}.$$

Dominance by a Mixed Strategy

Example

	X	Y	Z
A	$2, -$	$0, -$	$1, -$
B	$0, -$	$2, -$	$1, -$
C	$1, -$	$1, -$	$0, -$
D	$1, -$	$0, -$	$0, -$

- D is weakly dominated by A,
- C is weakly dominated by $\frac{1}{2} \cdot A + \frac{1}{2} \cdot B$,
- D is strictly dominated by $\frac{1}{2} \cdot A + \frac{1}{2} \cdot C$.

Iterated Elimination of Strategies

Consider weak dominance by a mixed strategy.

	X	Y	Z
A	$2, 1$	$0, 1$	$1, 0$
B	$0, 1$	$2, 1$	$1, 0$
C	$1, 1$	$1, 0$	$0, 0$
D	$1, 0$	$0, 1$	$0, 0$

- D is weakly dominated by A,
- Z is weakly dominated by X,
- C is weakly dominated by $\frac{1}{2} \cdot A + \frac{1}{2} \cdot B$.

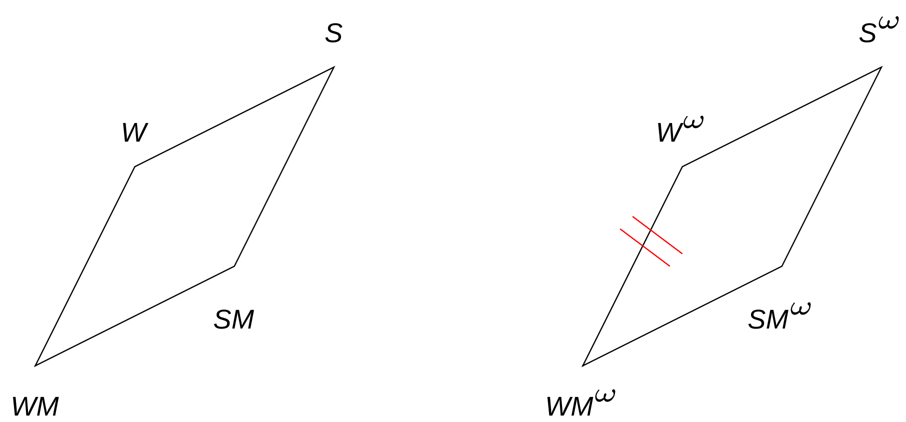
By eliminating them we get the final outcome:

	X	Y
A	$2, 1$	$0, 1$
B	$0, 1$	$2, 1$

Relative Strength of Strategy Elimination

- Weak dominance by a pure strategy is less powerful than weak dominance by a mixed strategy, but
- iterated elimination using weak dominance by a pure strategy (W^ω) can be more powerful than iterated elimination using weak dominance by a mixed strategy (MW^ω).

In general (Apt '07):



Best responses to Mixed Strategies

- s_i is a best response to m_{-i} if

$$\forall s'_i \in S_i \; p_i(s_i, m_{-i}) \geq p_i(s'_i, m_{-i}).$$

- $support(m_i) := \{a \in S_i \mid m_i(a) > 0\}.$
- Theorem (Pearce '84) In a 2-player finite game
  - s_i is strictly dominated by a mixed strategy iff it is not a best response to a mixed strategy.
  - s_i is weakly dominated by a mixed strategy iff it is not a best response to a mixed strategy with full support.

IESDMS

Theorem

- If G' is an outcome of IESDMS starting from G, then m is a Nash equilibrium of G' iff it is a Nash equilibrium of G.
- If G is solved by IESDMS, then the resulting joint strategy is a unique Nash equilibrium of G.
- (Osborne, Rubinstein, ’94) Outcome of IESDMS is unique (order independence).

IESDMS: Example

Beauty-contest game

- each set of strategies = $\{1, \ldots, 100\}$,
- payoff to each player:
  1 is split equally between the players whose submitted number is closest to $\frac{2}{3}$ of the average.
- This game is solved by IESDMS, in 99 steps.
- Hence it has a unique Nash equilibrium, $(1, \ldots, 1)$.

IEWDMS

Theorem

- If G' is an outcome of IEWDMS starting from G and m is a Nash equilibrium of G', then m is a Nash equilibrium of G.
- If G is solved by IEWDMS, then the resulting joint strategy is a Nash equilibrium of G.
- Outcome of IEWDS does not need to be unique (no order independence).
- Every mixed extension of a finite strategic game has a Nash equilibrium in which no pure strategy is weakly dominated by a mixed strategy.

Rationalizable Strategies

- Introduced in Bernheim '84 and Pearce '84.
- Strategies in the outcome of IENBRM.
- Subtleties in the definition ...

Theorem

- (Bernheim '84) If G' is an outcome of IENBRM starting from G, then m is a Nash equilibrium of G' iff it is a Nash equilibrium of G.
- If G is solved by IESDMS, then the resulting joint strategy is a unique Nash equilibrium of G.
- (Apt '05) Outcome of IENBRM is unique (order independence).

Pre-Bayesian Games

(Hyafil, Boutilier '04, Ashlagi, Monderer, Tennenholtz '06,)

- In a strategic game after each player selected his strategy each player knows all the payoffs (complete information).
- In a pre-Bayesian game after each player selected his strategy each player knows only his payoff (incomplete information).
- This is achieved by introducing (private) types.

Pre-Bayesian Games: Definition

Pre-Bayesian game for $n \geq 2$ players:

- (possibly infinite) set A_i of actions,
- (possibly infinite) set Θ_i of (private) types,
- payoff function $p_i : A_1 \times \ldots \times A_n \times \Theta_i \rightarrow \mathbb{R}$,

for each player i.

Basic assumptions:

- Nature moves first and provides each player i with a θ_i,
- players do not know the types received by other players,
- players choose their actions simultaneously,
- each player is rational (wants to maximize his payoff),
- players have common knowledge of the game and of each others' rationality.

Nash Equilibrium

- A strategy for player i:

$$s_i(\cdot) \in A_i^{\Theta_i}.$$

- Joint strategy $s_(\cdot)$ is a Nash equilibrium if each $s_i(\cdot)$ is a best response to $s_{-i}(\cdot)$:

$$\forall \theta \in \Theta \ \forall i \in \{1, \ldots, n\} \ \forall s_i'(\cdot) \in A_i^{\Theta_i}$$
$$p_i(s_i(\theta_i), s_{-i}(\theta_{-i}), \theta_i) \geq p_i(s_i'(\theta_i), s_{-i}(\theta_{-i}), \theta_i).$$

- Note: For each $\theta \in \Theta$ we have one strategic game. $s_(\cdot)$ is a Nash equilibrium if for each $\theta \in \Theta$ the joint action $(s_1(\theta_1), \ldots, s_n(\theta_n))$ is a Nash equilibrium in the θ-game.

Quiz

- $\Theta_1 = \{U, D\}$, $\Theta_2 = \{L, R\}$,
- $A_1 = A_2 = \{F, B\}$.

L

		F	B
U	F	$2, 1$	$2, 0$
	B	$0, 1$	$2, 1$

		F	B
D	F	$3, 1$	$2, 0$
	B	$5, 1$	$4, 1$

R

		F	B
U	F	$2, 0$	$2, 1$
	B	$0, 0$	$2, 1$

		F	B
D	F	$3, 0$	$2, 1$
	B	$5, 0$	$4, 1$

Which strategies form a Nash equilibrium?

Answer

- $\Theta_1 = \{U, D\}$, $\Theta_2 = \{L, R\}$,
- $A_1 = A_2 = \{F, B\}$.

L

U		F	B
	F	2, 1	2, 0
	B	0, 1	2, 1

D		F	B
	F	3, 1	2, 0
	B	5, 1	4, 1

R

U		F	B
	F	2, 0	2, 1
	B	0, 0	2, 1

D		F	B
	F	3, 0	2, 1
	B	5, 0	4, 1

- Strategies
  $s_1(U) = F, s_1(D) = B$,
  $s_2(L) = F, s_2(R) = B$
  form a Nash equilibrium.

But . . .

Nash equilibrium does not need to exist in mixed extensions of finite pre-Bayesian games.

Example: Mixed extension of the following game.

- $\Theta_1 = \{U, B\}$, $\Theta_2 = \{L, R\}$,
- $A_1 = A_2 = \{C, D\}$.

L, U:

	C	D
C	$2, 2$	$0, 0$
D	$3, 0$	$1, 1$

R, U:

	C	D
C	$2, 1$	$0, 0$
D	$3, 0$	$1, 2$

L, B:

	C	D
C	$1, 2$	$3, 0$
D	$0, 0$	$2, 1$

R, B:

	C	D
C	$1, 1$	$3, 0$
D	$0, 0$	$2, 2$

Intelligent Design

Finance & Economics

Economics focus

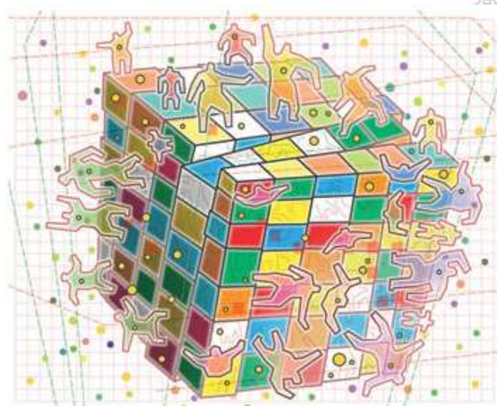
Intelligent design

Oct 18th 2007

From *The Economist* print edition

A theory of an intelligently guided invisible hand wins the Nobel prize

Jac



"WHAT on earth is mechanism design?" was the typical reaction to this year's Nobel prize in economics, announced on October 15th. In this era of "Freakonomics", in which everyone is discovering their inner economist, economics has become unexpectedly sexy. So what possessed the Nobel committee to honour a subject that sounds so thoroughly dismal? Why didn't they follow the lead of the peace-prize judges, who know not to let technicalities about being true to the meaning of the award get in the way of good headlines?

In fact, despite its dreary name, mechanism design is a hugely important area of economics, and underpins much of what dismal scientists do today. It goes to the heart of one of the biggest challenges in economics: how to arrange our economic interactions so that, when everyone behaves in a self-interested manner, the result is something we all like. The word "mechanism" refers to the institutions and the rules of the game that govern our economic activities, which can range from a Ministry of Planning in a command economy to the internal organisation of a company to trading in a market.

Intelligent Design

A theory of an intelligently guided invisible hand wins the Nobel prize

WHAT on earth is mechanism design? was the typical reaction to this year's Nobel prize in economics, announced on October 15th.
[...]
In fact, despite its dreary name, mechanism design is a hugely important area of economics, and underpins much of what dismal scientists do today. It goes to the heart of one of the biggest challenges in economics: how to arrange our economic interactions so that, when everyone behaves in a self-interested manner, the result is something we all like.
(The Economist, Oct. 18th, 2007)

Mechanism Design

Decision problem for n players:

- set D of decisions,
- for each player i a set Θ_i of (private) types Θ_i
- and a utility function

$$v_i : D \times \Theta_i \rightarrow \mathcal{R}.$$

- Intuition: $v_i(d, \theta_i)$ represents the benefit to player i of type θ_i from decision $d \in D$.
- When the individual types are $\theta_1, \ldots, \theta_n$ $\sum_{i=1}^{n} v_i(d, \theta_i)$ represents the social welfare from the decision $d \in D$.

Decision Rules

- Decision rule is a function

$$f : \Theta_1 \times \ldots \times \Theta_n \to D.$$

- Decision rule f is efficient if

$$\sum_{i=1}^{n} v_i(f(\theta), \theta_i) \geq \sum_{i=1}^{n} v_i(d, \theta_i)$$

  for all $\theta \in \Theta$ and $d \in D$.

- Intuition: f is efficient if it always yields a best decision for the society.

Set up

- each player i receives/has a type θ_i,
- each player i submits to the central authority a type θ_i',
- the central authority computes decision

$$d := f(\theta_1', \ldots, \theta_n'),$$

and communicates it to each player i.

Example 1: Sealed-Bid Auction

- $D = \{1, \ldots, n\}$,
- each Θ_i is $\mathbb{R}_+$,
- $v_i(d, \theta_i) := \begin{cases} \theta_i & \text{if } d = i \\ 0 & \text{otherwise} \end{cases}$
- argsmax $\theta := \mu i(\theta_i = \max_{j \in \{1,\ldots,n\}} \theta_j)$.
- $f(\theta) :=$ argsmax θ.
- Note: f is efficient.

Example 2: Public Project Problem

- c: cost of the public project (e.g., a bridge),
- $D = \{0, 1\}$,
- each Θ_i is $\mathbb{R}_+$,
- $v_i(d, \theta_i) := d(\theta_i - \frac{c}{n})$,
- $$f(\theta) := \begin{cases} 1 & \text{if } \sum_{i=1}^{n} \theta_i \geq c \\ 0 & \text{otherwise} \end{cases}$$
- Note: f is efficient.

Manipulations

An optimal strategy for player i in public project problem:

- if $\theta_i \geq \frac{c}{n}$ submit $\theta_i' = c$.
- if $\theta_i < \frac{c}{n}$ submit $\theta_i' = 0$.

Example $c = 30$

player	type
A	6
B	7
C	25

Revised Set-up: Direct Mechanisms

- each player i receives/has a type θ_i,
- each player i submits to the central authority a type θ'_i,
- the central authority computes decision

$$d := f(\theta'_1, \ldots, \theta'_n),$$

and taxes

$$(t_1, \ldots, t_n) := g(\theta'_1, \ldots, \theta'_n) \in \mathbb{R}^n,$$

and communicates to each player i both d and t_i.

- final utility function for player i:

$$u_i(d, \theta_i) = v_i(d, \theta_i) + t_i.$$

Groves Mechanisms

- $t_i(\theta') := \sum_{j \neq i} v_j(f(\theta'), \theta'_j) + h_i(\theta'_{-i})$,, where

  $h_i : \Theta_{-i} \rightarrow \mathbb{R}$ is an arbitrary function.

- Intuition:
  $\sum_{j \neq i} v_j(f(\theta'), \theta'_j)$
  is the social welfare with i excluded from decision $f(\theta')$.

- Note:

$$u_i((f, t)(\theta), \theta_i) = \sum_{j=1}^{n} v_j(f(\theta), \theta_j) + h_i(\theta_{-i}).$$

Groves Mechanisms, ctd

- Direct mechanism (f, t) is incentive compatible if for all $\theta \in \Theta$, $i \in \{1, \ldots, n\}$ and $\theta_i' \in \Theta_i$

$$u_i((f,t)(\theta_i, \theta_{-i}), \theta_i) \geq u_i((f,t)(\theta_i', \theta_{-i}), \theta_i).$$

- Theorem (Groves '73)
  Suppose f is efficient. Then each Groves mechanism is incentive compatible.

- Groves mechanism with tax function $t := (t_1, \ldots, t_n)$ is feasible if $\sum_{i=1}^{n} t_i(\theta) \leq 0$ for all θ.

Special Case: Pivotal Mechanism

- $h_i(\theta_{-i}) := -\max_{d \in D} \sum_{j \neq i} v_j(d, \theta'_j)$.
- Then

$$t_i(\theta') := \sum_{j \neq i} v_j(f(\theta'), \theta'_j) - \max_{d \in D} \sum_{j \neq i} v_j(d, \theta'_j) \leq 0.$$

- Conclusion Pivotal mechanism is feasible.

Example 1: Vickrey Auction

Vickrey auction: the winner pays the 2nd highest bid.

Example:

player	bid	tax to authority	util.
A	18	0	0
B	24	-21	3
C	21	0	0

Formally

θ^*: the reordering of θ is descending order,

$$f(\theta) := \text{argsmax}\,\theta,$$

$$t_i(\theta) := \begin{cases} -\theta^*_2 & \text{if } i = \text{argsmax}\,\theta \\ 0 & \text{otherwise} \end{cases}$$

Theorem (Vickrey '61): Vickrey auction is incentive compatible.

Example 2: Public Project Problem

Suppose $c = 30$ and $n = 3$.

player	type	tax	u_i
A	6	0	-4
B	7	0	-3
C	25	-7	8

player	type	tax	u_i
A	4	-5	-5
B	3	-6	-6
C	22	0	0

Formally

In the public problem

$$t_i(\theta) := \begin{cases} 0 & \text{if } \sum_{j \neq i} \theta_j \geq \frac{n-1}{n} c \text{ and } \sum_{j=1}^{n} \theta_j \geq c \\ \sum_{j \neq i} \theta_j - \frac{n-1}{n} c & \text{if } \sum_{j \neq i} \theta_j < \frac{n-1}{n} c \text{ and } \sum_{j=1}^{n} \theta_j \geq c \\ 0 & \text{if } \sum_{j \neq i} \theta_j \leq \frac{n-1}{n} c \text{ and } \sum_{j=1}^{n} \theta_j < c \\ \frac{n-1}{n} c - \sum_{j \neq i} \theta_j & \text{if } \sum_{j \neq i} \theta_j > \frac{n-1}{n} c \text{ and } \sum_{j=1}^{n} \theta_j < c \end{cases}$$

Relation to pre-Bayesian Games

- Strategy $s_i(\cdot)$ is dominant if for all $a \in A$ and $\theta_i \in \Theta_i$

$$\forall a \in A \; p_i(s_i(\theta_i), a_{-i}, \theta_i) \geq p_i(a_i, a_{-i}, \theta_i).$$

- A pre-Bayesian game is of a revelation-type if $A_i = \Theta_i$ for all $i \in \{1, \ldots, n\}$.
- So in a revelation-type pre-Bayesian game the strategies of player i are the functions on Θ_i.
- A strategy for player i is called truth-telling if it is the identity function $\pi_i(\cdot)$.

Relation to pre-Bayesian Games, ctd

- Mechanism design (as discussed here) can be viewed as an instance of the revelation-type pre-Bayesian games.
- With each direct mechanism (f, t) we can associate a revelation-type pre-Bayesian game:
  - Each Θ_i as in the mechanism,
  - Each $A_i = \Theta_i$,
  - $p_i(\theta_i', \theta_{-i}, \theta_i) := u_i((f, t)(\theta_i', \theta_{-i}), \theta_i)$.
- Note Direct mechanism (f, t) is incentive compatible iff in the associated pre-Bayesian game for each player truth-telling is a dominant strategy.
- Conclusion In the pre-Bayesian game associated with a Groves mechanism, $(\pi_1(\cdot), \ldots, \pi_i(\cdot))$ is a dominant strategy Nash equilibrium.